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NON-LOCAL SOLUTIONS TO DYNAMIC EQUILIBRIUM MODELS: THE APPROXIMATE STABLE MANIFOLDS APPROACH

CONTENTS

ABSTRACT

This paper presents a method to construct a sequence of approximate policy functions of increasing accuracy on non-local domains. The method is based upon the notion of stable manifold originated from dynamical systems theory. The approximate policy functions are constructed employing the contraction mapping theorem and the fact that solutions to rational expectations models converge to a steady state. The approach allows us to derive the accuracy of the approximations and their domain of definition. The method is applied to the neoclassical growth model and compared with the perturbation method. Just the second approximation of the proposed approach yields very high accuracy of the approximate solution on a global domain. In contrast to the Taylor series expansions, the solutions of the method inherit globally the properties of the true solution such as monotonicity and concavity.

Keywords: dynamic equilibrium, rational expectations, non-linear perfect foresight models, stable manifold, perturbation method, extended path, neoclassical growth model

JEL codes: C62, C63, D9, D58

NON-TECHNICAL SUMMARY

General equilibrium models are intrinsically non-linear. Utility functions that reflect risk aversion, production functions with diminishing return to scale, adjustment costs in investment are just a few examples of non-linearities presented in the models. Most of the general equilibrium models do not admit a closed-form solution; the only possibility is to find an approximate solution using numerical methods. In macroeconomic literature, linearised versions of DSGE models have been most widely used primarily because of their simplicity and possibility to estimate the models employing standard econometric tools such as the Kalman filter. However, non-linearities play a crucial role in many important problems that cannot be solved by linear models. These problems include the macroeconomic determinants of risk premium, consequences of uncertainty shocks, time-varying variances in macro-models, optimal policy, asymmetric responses to positive and negative shocks, threshold effect, occasionally binding constraints etc. (see Amisano and Tristani (2007) and (2011), Fernández-Villaverde et al. (2011), Binsbergen et al. (2012), Kim and Kim (2007), and Kollmann (2002) and (2004), Judd (1998), Woodford (2003), and Gavin et al. (2013)).

There are two types of basic methods commonly used to solve non-linear rational expectations models: local (perturbation) and global (projection, stochastic simulation etc.). Each type has its own advantages and disadvantages. The global methods can compute solutions on large domains as opposed to the perturbation methods which are valid only around a steady state. However, the global methods suffer from computational costs growing fast with the dimensionality of state space. This phenomenon, called the curse of dimensionality, restricts the application of the projection methods even to medium-size models.

Perturbation methods solve the coefficients of Taylor series expansions of the true model solution around a deterministic steady state. Compared with the global methods, the perturbation methods have three key advantages: higher computational speed, the ease with which they can be applied to models with a large number of state variables, and the development of user-friendly software like Dynare. However, higher-order perturbation approximations are polynomials, and, as a consequence, they do not inherit global properties, such as monotonicity and convexity, from the true underlying policy function. Another unfavourable feature of polynomials related to the already mentioned ones is that they generate unbounded solutions if an economy is far away from the steady state (Den Haan and de Wind (2009) and (2012)). This situation can occur when an economy (a) undergoes a big shock; (b) undergoes a sequence of persistent shocks; (c) is one in transition or emerging for which its current position can be at a distance from the steady state.

For the second-order Taylor series expansions, Kim et al. (2008) have proposed a method, which they call pruning, to address the problem of explosive sample paths. The pruning procedure replaces the quadratic term with cross-products of the first-order solution and thus generates a recursively linear solution. Kim et al. (2008) show that the pruned approximation does not explode. Various extensions and modifications of the pruning methods are proposed by Lombardo (2010), Den Haan and De Wind (2012), and Andreasen et al. (2013). Nevertheless, under a sufficiently large shock the pruning procedure may provide a first few impulse responses with wrong signs, i.e. instead of being positive, they could take negative values. This case

seems even worse than the explosive dynamics since the impulse responses for a first few periods are the most interesting and relevant for theoretical implications of a model as well as a policy analysis; therefore, their incorrect signs could mislead a researcher or a policymaker. In fact, the pruning procedure results in convergence to zero of the impulse responses function. This fact by itself does not give much information about the model's properties such as the features of the variable interrelation, shock propagation etc.

This paper presents a general method to construct a sequence of policy functions of increasing accuracy on non-local domains. The proposed method is based upon a concept originated from dynamical systems theory (Galor (2007), Grandmont (2008)), which is called a stable manifold, i.e. the set of points that approach the saddle point as time tends to infinity. In fact, the set of solutions to a non-linear rational expectations model determines the stable manifold because each solution must satisfy the stability condition, i.e. the convergence to the steady state in the long-run. In economic literature, this set is represented by a graph of a policy function (or, in other words, a decision function) that maps the state variables into the control variables.

The algorithm implies an iterative procedure which is known as the method of successive approximations. This method can easily be implemented and incorporated into existing software platforms such as Dynare (Adjemian et al. (2011)). The approach also allows for deriving the estimate of the accuracy and domain of the approximate solutions. Compared with the perturbation methods, the proposed solutions are non-local and, by construction, exponentially stable; therefore, they cannot explode.

The presented method is applied to the deterministic neoclassical growth model (Brock and Mirman (1972)). We compare the accuracy of the approximate solution with the Taylor series expansions and find the following advantages of the proposed method:
(i) just the first approximation of the algorithm gives a very high global approximation accuracy;
(ii) the second- and third-order approximate solutions are almost indistinguishable globally from the true solution;
(iii) even within the domain of convergence of the Taylor series expansion, the third-order approximate solution is more accurate than the 16th-order Taylor series expansion;
(iv) unlike the Taylor approximation, the proposed method inherits globally the properties of the true solution such as monotonicity and concavity.

The proposed method relates to the extended path method (Fair and Taylor (1983)). Namely, at each point in time the solution of the extended path method applied to the transformed system is equal to the value of the corresponding approximate policy function at the corresponding time. Therefore the presented approach may be treated as a rigorous proof of the convergence of the extended path method. Using Newton's method instead of mapping iterations may significantly accelerate the convergence of the computation process.

Although this paper primarily focuses on non-linear deterministic rational expectation models (the perfect foresight models), we also outline possible extensions of the deterministic case to the stochastic one. The analogy with the extended path method would suggest that the approach, which is proposed by Adjemian and Juillard (2010) and called the stochastic extended path approach, can also be applied in our case. This approach implies that the conditional expectations are computed employing either quadratures or some stochastic simulation algorithms. Another feature of the method is its capability to handle non-differentiable problems such as occasionally binding constraints (the zero lower bound problem, models with heterogeneous agents and constraints on the financial assets available to agents etc.).

1. INTRODUCTION

Perturbation and projection methods are commonly used to solve non-linear dynamic equilibrium models. The projection methods can compute solutions on large domains as opposed to the perturbation methods which are valid only around a steady state.

However, the projection methods suffer from computational costs growing fast with the dimensionality of state space. This phenomenon, called the curse of dimensionality, restricts the application of the projection methods even to medium-size models.

The perturbation methods solve the coefficients of the Taylor expansions of the true model solution around a deterministic steady state[1]. Compared with the projection methods, the perturbation methods have higher computational speed, and can be easily applied to models with a large number of state variables. However, the higher-order perturbation approximations do not inherit global properties such as monotonicity and convexity from the true underlying policy function since they are polynomials. As a result, for sufficiently large shocks (or initial conditions) the approximated solution can imply explosive dynamics, even if the original system is still stable for the same shocks (Kim et al. (2008), and Den Haan and De Wind (2009) and (2012)).

This problem gets worsened by the fact that it is not known a priori whether a shock is sufficiently large or not. For example, Figure 1 illustrates the impulse response functions for inflation and output following a loan-to-value ratio shock. Simulations are done using the second-order approximation for a DSGE model of Latvia with a banking sector (Ajevskis and Vītola (2011)). The standard deviation of the shock equals 0.03. This value does not look very large because the steady state value of the loan-to-value ratio is equal to 0.70. However, as Figure 1 shows, after 15 quarters inflation and output take the values of 23% and 18% respectively; these numbers are already fairly high and must be treated cautiously. Then, after 16 quarters the variables take absolutely unreasonable magnitudes that exceed 10^6% (not shown in Figure 1).

For the second-order Taylor series expansions, Kim et al. (2008) have proposed a method, which they call pruning, to address the problem of explosive sample paths. The pruning procedure replaces the quadratic term with cross-products of the first-order solution and thus generates a recursively linear solution. Kim et al. (2008) show that the pruned approximation does not explode. Various extensions and modifications of the pruning methods are proposed by Lombardo (2010), Den Haan and De Wind (2009) and (2012), and Andreasen et al. (2013). Nevertheless, it is easily shown that under a sufficiently large shock the pruning procedure may provide a first few impulse responses with wrong signs. This case seems even worse than the explosive dynamics since the impulse responses for a first few periods are

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[1] Several algorithms have been proposed for solving linear models: for instance, Blanchard and Kahn (1980), Uhlig (1999), Anderson and Moor (1985), Klein (2000), Sims (2001), and Christiano (2002). Higher order perturbation methods are studied in Judd (1998), Jin and Judd (2002), Collard and Juillard (2001), Schmitt-Grohé and Uribe (2004), Swanson et al. (2005), Kim et al. (2008), Andreasen et al. (2013), and Gomme and Klein (2011).

the most interesting for theoretical implications of a model as well as a policy analysis; therefore, their incorrect signs could mislead a researcher or a policymaker. In fact, the pruning procedure results in convergence to zero of the impulse responses function. This fact by itself does not give much information about the model's properties such as the features of the variable interrelation, shock propagation etc.

Figure 1. The impulse response functions for inflation and output to the loan-to-value ratio shock in a DSGE model of Latvia with a banking sector
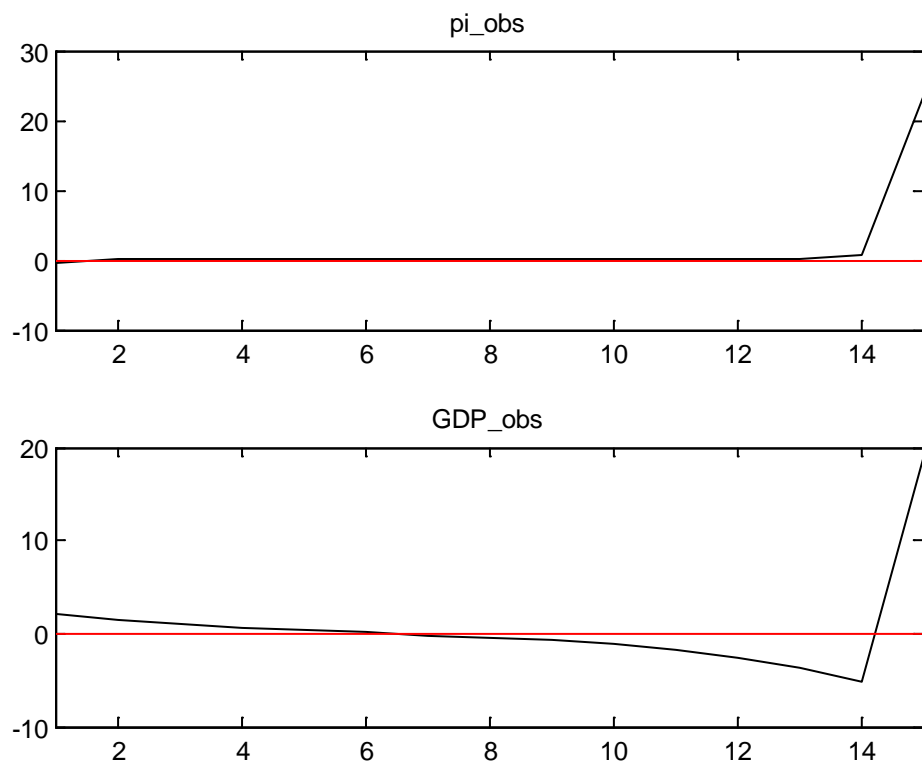
This paper presents a general method to construct a sequence of policy functions of increasing accuracy on a non-local domain. The proposed method is based upon a concept originated from dynamical systems theory (Galor (2007), Grandmont (2008)), which is called a stable manifold, i.e. the set of points that approach the saddle point as time tends to infinity. In fact, the set of solutions to a non-linear rational expectations model determines the stable manifold because each solution must satisfy the stability condition, i.e. the convergence to the steady state in the long-run. In economic literature, this set is represented by a graph of a policy function (in other words, a decision function) that maps the state variables into the control variables.

Initially, the proposed method involves the same steps as the perturbation methods: (a) find a steady state; (b) linearise the model around the steady state; and (c) decompose the Jacobian matrix at the steady state into stable and unstable parts. The next step is to project the original system on the stable eigenspace (spanned on the stable eigenvectors) and the unstable one (spanned on the unstable eigenvectors). As a result, the system will be presented by two subsystems interrelated only through

non-linear terms. These non-linear terms are obtained as residuals after subtraction of the linearised system from the original one; hence these terms vanish, together with their first derivatives, at the origin. Such a transformation makes the obtained system convenient to take the next stage of the method. Specifically, the approximate solutions are constructed by employing (i) the convergence of solutions to the steady state; and (ii) the contraction mapping theorem (Ljungqvist and Sargent (2004)).

The approach allows us to derive the estimate of the accuracy and domain of the approximate solutions. The accuracy and domain depend on the magnitude and norm of the Jacobian matrix of the transformed system and the spectral properties of the linear part (the modulus of the largest stable eigenvalue and the modulus of the smallest unstable eigenvalue). In some cases (for instance, the neoclassical growth model considered as an example in Section 4), already the first approximation in the constructed sequence of solutions gives a very high accuracy on a global domain.

Compared with the perturbation methods, the proposed solutions are non-local and, by construction, exponentially stable; therefore, they cannot explode. In contrast to the projection methods, the presented algorithm finds the policy function only at a number of the forward solution points rather than the whole domain of definition. This feature implies less time-consuming computations compared with those of projection methods. Such an approach is preferable to compute a particular perfect foresight solution, for example, a solution with specific initial conditions (a specific impulse response function).

The proposed method relates to the extended path method (Fair and Taylor (1983)). Namely, at each point in time the solution of the extended path method applied to the transformed system is equal to the value of the corresponding approximate policy function at the corresponding time. This way the presented approach may be treated as a rigorous proof of convergence of the extended path method. Using Newton's method instead of mapping iterations may significantly accelerate the convergence of the computation process. However, Newton's method imposes higher requirements on the degree of smoothness for mappings; furthermore, the domain where the algorithm converges may be smaller.

Although this paper focuses primarily on non-linear deterministic rational expectation models (the perfect foresight case), we also outline the extension of the deterministic case to the stochastic one. Theoretically, this can be done by the use of the implicit function theorem under the condition of sufficiently small stochastic innovations (see Jinn and Judd (2002)). Practically, the analogy with the extended path method would suggest that the approach, which is proposed by Adjemian and Juillard (2010) and called the stochastic extended path approach, can also be applied in our case. This approach implies that the conditional expectations are computed employing either quadratures or some stochastic simulation algorithms.

The presented method is applied to the log preference and complete depreciation case of the deterministic neoclassical growth model (Brock and Mirman (1972)). We compare the accuracy of the approximate solution with the Taylor series expansions and find the following advantages of the proposed method:

(i) already the first approximation of the algorithm gives a very high global approximation accuracy;

(ii) the second- and third-order approximate solutions are almost indistinguishable from the true solution globally;

(iii) even within the domain of the Taylor series expansion convergence, the third-order approximate solution of the implicit scheme is more accurate than the 16th-order Taylor series expansion;

(iv) unlike the Taylor approximation, the proposed method inherits globally the properties of the true solution such as monotonicity and concavity.

Another feature of the method is its capability to handle non-differentiable problems such as occasionally binding constraints (the zero lower bound problem, models with heterogeneous agents and constraints on the financial assets available to agents etc.). This feature results from the fact that the contraction mapping theorem requires a less restrictive condition than differentiability: namely, this condition is Lipschitz continuity which means that the slope of the mappings involved must be bounded.

2. THE MODEL

DSGE models usually take the following form:

$$E_t f(y_{t+1}, y_t, x_{t+1}, x_t, z_t, \varepsilon_{t+1}) = 0 \quad (1),$$

$$z_{t+1} = \Lambda z_t + \sigma \varepsilon_{t+1}, \ \varepsilon_t \sim N(0, \sigma\Omega) \ (2)$$

where E_t denotes the conditional expectations operator; x_t is an $n_x x1$ vector of endogenous state variables at time t (e.g. capital and lagged variables); y_t is an $n_y x1$ vector containing t-period endogenous variables that are not state variables (e.g. consumption, labour, prices, Lagrange multipliers); z_t is an $n_z x1$ vector of exogenous state (random) variables at time t (e.g. productivity); ε_{t+1} is a vector of disturbances; $\sigma\Omega$ is $n_z x n_z$ covariance matrix of disturbances; f maps $R^{n_y} \times R^{n_y} \times R^{n_x} \times R^{n_x} \times R^{n_z} \times R^{n_z}$ into $R^{n_y} \times R^{n_x}$ and is assumed to be at least 1-time continuously differentiable. The scalar σ is a scaling parameter for the disturbance terms ε_t. All eigenvalues of the matrix Λ have modulus less than one.

2.1 Deterministic Model

In this paper, we shall primarily concern ourselves with the deterministic perfect foresight equilibrium, i.e. the case where $\sigma = 0^2$; then the system (1) – (2) takes the form

$$f(y_{t+1}, y_t, x_{t+1}, x_t, z_t, 0) = 0 \quad (3),$$

$$z_{t+1} = \Lambda z_t \,(4).$$

Here z_t may be treated as a vector of initial values of temporary shocks at time t. In the sequel, we omit the last argument of f. The solution to the model (3) and (4) is of the form:

$$y_t = \tilde{h}(x_t, z_t) \ (5),$$

$$x_{t+1} = \tilde{g}(x_t, z_t) \ (6)$$

where $\tilde{h}$ maps $R^{n_x} \times R^{n_z}$ into R^{n_y} and is called the policy function; and $\tilde{g}$ maps $R^{n_x} \times R^{n_z}$ into R^{n_x}.

2.2 Transformation of the Model

We define the deterministic steady state as vectors $(\bar{y}, \bar{x}, 0)$ such that

$$f(\bar{y}, \bar{y}, \bar{x}, \bar{x}, 0) = 0\,(7).$$

An additional condition imposed on the solution (6) is that x_t must tend to the steady state $\bar{x}$ as $t \to \infty$; the variable z_t already satisfies the condition because of the matrix Λ property.

By $(\hat{y}_t, \hat{x}_t)$, we denote the vector of deviation from the steady state. Linearising (3) around the steady state, we have

$$f_1 \hat{y}_{t+1} + f_2 \hat{y}_t + f_3 \hat{x}_{t+1} + f_4 \hat{x}_t + f_5 z_t + N(\hat{y}_{t+1}, \hat{y}_t, \hat{x}_{t+1}, \hat{x}_t, z_t) = 0\,(8)$$

where f_i, $i = 1 \div 5$, are partial derivatives of the mapping f with respect to y_{t+1}, y_t, x_{t+1}, x_t, z_t, respectively at the point $(\bar{y}, \bar{y}, \bar{x}, \bar{x}, 0)$, and N is defined by

$$N(\hat{y}_{t+1}, \hat{y}_t, \hat{x}_{t+1}, \hat{x}_t, \hat{z}_t) = f(\bar{y} + \hat{y}_{t+1}, \bar{y} + \hat{y}_t, \bar{x} + \hat{x}_{t+1}, \bar{x} + \hat{x}_t, z_t) -$$
$$- f_1 \hat{y}_{t+1} - f_2 \hat{y}_t - f_3 \hat{x}_{t+1} - f_4 \hat{x}_t - f_5 z_t$$

The mapping N will be referred to as the non-linear part of f. By the assumption on f, the mapping N is continuously differentiable and vanishes, together with its first derivatives, at $(0,0,0,0,0)$. For the sake of simplicity, we assume that equation (8) can be transformed in such a way that the mapping N does not depend on $\hat{y}_{t+1}$ and $\hat{x}_{t+1}$. This transformation can be done for many DSGE models (see Appendix A for

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[2] In Section 3, we consider the case where $\sigma > 0$.

the neoclassical growth model); otherwise we assume that the implicit function theorem allows us to obtain the following representation of (8):

$$f_1 \hat{y}_{t+1} + f_2 \hat{y}_t + f_3 \hat{x}_{t+1} + f_4 \hat{x}_t + f_5 z_t + N(\hat{y}_t, \hat{x}_t, z_t) = 0 \ (9).$$

The equations (4) and (9) can be written in the vector form as:

$$\Phi w_{t+1} = \Gamma w_t + \begin{pmatrix} 0 \\ N(w_t) \end{pmatrix} (10)$$

where $w_t = (z_t, \hat{x}_t, \hat{y}_t)'$, $\Phi = \begin{pmatrix} I & O & O \\ O & f_3' & f_1' \end{pmatrix}$ and $\Gamma = \begin{pmatrix} \Lambda & 0 & 0 \\ f_5' & f_4' & f_2' \end{pmatrix}$.

We assume that the matrix Φ is invertible.[3] Note that the matrix Φ is quadratic with dimension $n_z + n_x + n_y$. Then multiplying both sides of (10) by Φ^{-1}, we get

$$w_{t+1} = K w_t + N_2(w_t) \cdot (11)$$

where

$$w_t = \begin{pmatrix} z_t \\ \hat{x}_t \\ \hat{y}_t \end{pmatrix}, \ K = \Phi^{-1} \Gamma = \begin{pmatrix} \Lambda & 0 & 0 \\ [f_3', f_1']^{-1} f_5' & [f_3', f_1']^{-1} f_4' & [f_3', f_1']^{-1} f_2' \end{pmatrix}$$

and

$$N_2(w_{t+1}) = \begin{pmatrix} 0 \\ \Phi^{-1} N_1(\hat{y}_t, \hat{x}_t, z_t) \end{pmatrix} = \begin{pmatrix} 0 \\ [f_3', f_1']^{-1} N_1(\hat{y}_t, \hat{x}_t, z_t) \end{pmatrix}.$$

Next, the matrix L is transformed into the Jordan canonical form[4]

$$K = ZJZ^{-1} \ (12)$$

where $J = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. The matrix A comprises the Jordan boxes having eigenvalues of modulus smaller than one, whereas the matrix B comprises the Jordan boxes having eigenvalues of modulus larger than one. Now we introduce the following new variables:

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[3] This assumption is made for ease of exposition. If Φ is a singular matrix, then in the sequel we must use a generalised eigenvalue decomposition instead of a simple one (King and Watson (2002)).

[4] The results below do not change if one uses the block-diagonal Schur factorisation in (12) rather than the Jordan decomposition. The block-diagonal Schur factorisation of a matrix K can be written in the form: $K = TMT^{-1}$, where M is two-block diagonal matrix, each diagonal block is a quasi upper-triangular Schur matrix corresponding to either stable or unstable eigenvalues and T is an invertible matrix. The function bdschur of Matlab Control System Toolbox performs this factorisation.

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = Z^{-1} \begin{pmatrix} z_t \\ \hat{x}_t \\ \hat{y}_t \end{pmatrix} \ (13).$$

Pre-multiplying (11) by Z^{-1}, we have

$$\begin{aligned} u_{t+1} &= Au_t + F(u_t, v_t) \\ v_{t+1} &= Bv_t + G(u_t, v_t) \end{aligned} \ (14)$$

where $u_t \in R^{n_z + n_x}$, $v_t \in R^{n_y}$, and

$$\begin{pmatrix} F(u_t, v_t) \\ G(u_t, v_t) \end{pmatrix} = Z^{-1} N_2 (Z \cdot [u_t, v_t]') .$$

By construction, it follows that

F(0,0) = 0, G(0,0) = 0, F'(0,0) = 0, G'(0,0) = 0 (15)

where F' and G' stand for the Jacobian matrix of F and G respectively, at the point (0,0).

2.3 Stable Manifold

By assumption, the matrix K is hyperbolic, i.e. its spectrum is disjoint from the unit circle. Therefore, a fundamental result from dynamical systems theory, which is called the stable manifold theorem (Galor (2007), Grandmont (2008)), can be applied to the system (14). The stable manifold is an invariant set of points that approach the saddle point as time tends to infinity. A set is invariant if it is invariant under the action of the dynamical system. The stable manifold theorem asserts that in a small neighbourhood of a fixed hyperbolic point there exists a unique stable invariant manifold with the same dimension as the stable linear subspace of the matrix K from (12). Moreover, the stable manifold can be represented as the graph of a function $h: U \rightarrow V$, where U and V are some neighbourhoods of the fixed point in the stable and unstable linear subspaces respectively. Another property of the stable manifold is that the tangent space to it at the fixed point is the stable linear subspace. In the case of rational expectations models or DSGE models in a broader sense, the graph of a policy function (5) is an example of the stable manifold.

The stable manifold theorem is typically proved locally, i.e. in the immediate vicinity of a fixed point (Hartmann (1982), and Katok and Hasselblatt (1995)). This paper proposes a recurrent procedure described in Subsection 2.5 that constructs the approximate stable manifolds directly and has a clear intuition. The proposed approach also allows for controlling the accuracy and domain of definition for approximate solutions.

2.4 Notation and Definition

This Section introduces some notation that will be necessary further on. By U_{r_u} and V_{r_v}, we denote closed balls of radii r_u and r_v centred at the origin of $R^{n_z+n_x}$ and R^{n_y} respectively. Let $X_{r_u,r_v} = U_{r_u} + V_{r_v}$ be the direct sum of these balls. By $|\cdot|$, we denote the Euclidean norm in $\Re^n$. The induced norm for real matrices is defined by

$$\|B\| = \sup_{|y|=1} |By|.$$

The matrix Z in (15) can be chosen in such a way that

$$\|A\| < \alpha + \gamma < 1 \text{ and } \|B^{-1}\| < \beta + \gamma < 1 \quad (16)$$

where α and β are the largest eigenvalue moduli of the matrixes A and B^{-1} respectively, and γ is arbitrarily small (Hartman (1982)). Note also that $\|B^{-1}\| < 1$ for γ sufficiently small. By definition, put

$$\|G'\|_{X_{r_u,r_v}} = \sup_{(u,v)\in X_{r_u,r_v}} \|G'(u,v)\| \text{ and } \|h_1'\| = \sup_{u\in U_{r_u,1}} \|h_1'(u)\|$$

where $G'(u,v)$ and h_1' are the Jacobian matrixes of the mappings $G(u,v)$ and $h_1(u)$ at the points (u,v) and u respectively. Define the norms in the Banach space of all continuous functions on X_{r_u,r_v} and U_r as

$$\|G\|_{X_{r_u,r_v}} = \sup_{(u,v)\in X_{r_u,r_v}} |G(u,v)| \text{ and } \|h\|_{U_r} = \sup_{u\in U_r} |h(u)|$$

respectively; and the norms in the Banach space of continuous maps on X_{r_u,r_v} and U_r into real matrixes space with the induced norm (16) as

$$\|G'\|_{X_{r_u,r_v}} = \sup_{(u,v)\in X_{r_u,r_v}} \|G'(u,v)\|, \quad \|F'\|_{X_{r_u,r_v}} = \sup_{(u,v)\in X_{r_u,r_v}} \|F'(u,v)\| \text{ and } \|h'\|_{U_r} = \sup_{u\in U_r} \|h'(u)\|$$

respectively, where $G'(u, v)$, $F'(u, v)$ and $h'(u)$ are the Jacobian matrixes of $G(u,v)$, $F(u, v)$ and $h(u)$ at (u, v) and u respectively.

Definition. A mapping $s: X \to Y$ is called Lipschitz continuous if there exists a real constant $l \geq 0$ such that for all x_1 and x_2 in X

$$|s(x_2) - s(x_1)| \leq l|x_2 - x_1|.$$

The smallest constant l is referred to as the Lipschitz constant for the mapping s.

It is easy to see that the constant

$$L = \max(\|G'\|_{X_{r_u,r_v}}, \|F'\|_{X_{r_u,r_v}}) \quad (17)$$

is an upper bound for the Lipschitz constant of the mappings G and F.

2.5 Approximating the Solution

Let $n \in N$ be bigger than one. From the second equation of (14) and invertibility of the matrix B it follows that

$$v_{t+n.} = -B^{-1}G(v_{t+n.}, v_{t+n.}) + B^{-1}v_{t+n.+1} \ (18).$$

Owing to the stability property, all solutions lying on a stable manifold satisfy $\lim_{n\to\infty}(u_{t+n}, h(u_{t+n})) = 0$. As stable linear subspace is tangent to the stable manifold at the origin, the mapping h vanishes to order 2; hence the magnitude of the mapping $h(u_{t+n+1})$ will be negligibly small if n is sufficiently large, even in comparison with the length of the vector u_{t+n+1}, which is also small. Such a judgement allows for omitting the term $B^{-1}v_{t+n.+1}$ on the right-hand side of (18). This results in the following equation:

$$v_{t+n} = -B^{-1}G(u_{t+n}, v_{t+n}) = T_{1,u_{t+n}}(v_{t+n}) \ (19)$$

where $T_{1,u}$ is a parameterised mapping of $V_{r_{v,0}}$ to $\Re^{n_y}$. As $G(0,0) = 0$ and $G'(0,0) = 0$, it follows that the contraction mapping theorem (Ljungqvist and Sargent (2004)) can be applied to $T_{1,u}$ for each $u_{t+n} \in U_{r_{u,0}}$ for some $U_{r_{u,0}}$. Therefore, there exists a fixed point h_1 of $T_{1,u}$ such that

$$h_1(u) = -B^{-1}G(u, h_1(u)), \ u \in U_{r_{u,0}} \ (20).$$

The condition for the graph of the mapping h to be an invariant manifold is that the image under transformation (14) of a general point of the graph of h must again be in the graph of h. This holds if and only if
$Bh(u)+G(u,h(u)) = h(Au+F(u,h(u)))$.

Taking into account the invertibility of B, we have

$h(u)=-B^{-1}G(u,h(u)) +B^{-1}h(Au+F(u,h(u)))(21)$.

To estimate the error of the approximation h_1, subtract the last equation from (20) at the point $x = x_{t+n}$

$$\left\| J'_{i+1,u_{t+n-i-1}} \right\| < 1 \ (22).$$

Taking the norms in (22), using the definition of $\left\| G' \right\|_{X_{r_{u,0},r_{v,0}}}$ and the triangle inequality, gives

$$\left| h_1(u_{t+n}) - h(u_{t+n}) \right| \le \left\| B^{-1} \right\| \left\| G' \right\|_{X_{r_{u,0},r_{v,0}}} \left| h_1(u_{t+n}) - h(u_{t+n}) \right| + \left\| B^{-1} \right\| \cdot \left| h(Au_{t+n} + F(u_{t+n}, h(u_{t+n}))) \right|$$

The neighbourhood $X_{r_{u,0},r_{v,0}}$ can be chosen such that

$$1 - \left\| B^{-1} \right\| \left\| G' \right\|_{X_{r_{u,0},r_{v,0}}} > 0 ;$$

therefore,

$$\left| h_1(u_{t+n}) - h(u_{t+n}) \right| \le (1 - \left\| B^{-1} \right\| \cdot \left\| G' \right\|_{X_{r_{u,0},r_{v,0}}})^{-1} \cdot \left\| B^{-1} \right\| \cdot \left| h(Au_{t+n} + F(u_{t+n}, h(u_{t+n}))) \right| (23).$$

From the inequality (23), it follows that there are three factors that determine the accuracy of the approximate solution h_1. The first factor is the magnitude of the norm $|h(\cdot)|$. Since the mapping h vanishes to order 2 at the origin, it follows that in some neighbourhood the magnitude of the norm of h will be smaller than the magnitude of the norm of argument $|u|$. Suppose, for example, that the domain $D = \{u: |u| \le \tau\}$, where $0 < \tau < 1$, is sufficiently small; then the norm $|h(u)|$ will have the second order of smallness in D, i.e. $|h(u)| \le C \tau^2$ for $u \in D$ where C is some constant.

The norm $h(\cdot)|$ would be even smaller if we take into account the second factor – the stability property, i.e. $\lim_{n \to \infty}(u_{t+n}, h(u_{t+n})) = 0$ for the solutions lying on the stable manifold.

Owing to the conditions $F(0,0) = 0$ and $F'(0,0) = 0$, the norm of $F(u_{t+n}, h(u_{t+n}))$ is very small; consequently,

$$|Au_{t+n} + F(u_{t+n}, h(u_{t+n}))| \approx |Au_{t+n}| \le \|A\| \cdot |u_{t+n}| \le \|A\| \cdot \tau.$$

Recall that the norm $\|A\| < 1$ and is determined by the maximum eigenvalue modulus of the matrix A. Therefore, the estimate of the distance between h and h_1 at the point u_{t+n} is

$$|h_1(u_{t+n}) - h(u_{t+n})| \le C\|A\|^2 \cdot \tau^2.$$

The third factor is the norm $\|B^{-1}\|$ that is determined by the minimum eigenvalue modulus of the unstable matrix B and indicates the degree of the instability of the system. Basically, the approximate solution h_1 is accurate in the domains $X_{r_{u,0}, r_{v,0}}$ where the inequality

$$\left(1 - \|B^{-1}\|\|G'\|_{X_{r_{u,0}, r_{v,0}}}\right)^{-1}\|B^{-1}\| < 1$$

holds and the norm of the true solution $|h(u_{t+n+1})|$ is small. These domains may be non-local because the mappings G and h have the property to vanish together with their first derivatives at the origin.

To obtain the next approximation, rewrite (20) for the time $t = n - 1$

$$v_{t+n-1} = -B^{-1}G(u_{t+n-1}, v_{t+n-1}) + B^{-1}v_{t+n} \quad (24).$$

Now substituting the approximation of h_1 (20) for v_{t+n} in (24) yields

$$v_{t+n-1} = -B^{-1}G(u_{t+n-1}, v_{t+n-1}) + B^{-1}h_1(Au_{t+n-1} + F(u_{t+n-1}, v_{t+n-1})).$$

The same way as for h_1, using the contraction mapping theorem gives the existence of a unique mapping h_2 such that

$$h_2(u_{t+n-1}) = -B^{-1}G(u_{t+n-1}, h_2(u_{t+n-1})) + B^{-1}h_1(Au_{t+n-1} + F(u_{t+n-1}, h_2(u_{t+n-1}))) \quad (25).$$

To find the accuracy of the approximation h_2, subtract (21) from (25)

$$h_2(u_{t+n-1}) - h(u_{t+n-1}) = -B^{-1}[G(u_{t+n-1}, h_1(u_{t+n-1})) - G(u_{t+n-1}, h(u_{t+n-1}))] -$$
$$- B^{-1}[h_1(Au_{t+n-1} + F(u_{t+n-1}, h_1(u_{t+n-1}))) - h(Au_{t+n-1} + F(u_{t+n-1}, h(u_{t+n-1})))]$$

Adding and subtracting $h_1(Au_{t+n-1} + F(u_{t+n-1}, h(u_{t+n-1})))$ on the right-hand side, yields

$$h_2(u_{t+n-1}) - h(u_{t+n-1}) = -B^{-1}[G(u_{t+n-1}, h_1(u_{t+n-1})) - G(u_{t+n-1}, h(u_{t+n-1}))] -$$
$$- B^{-1}[h_1(Au_{t+n-1} + F(u_{t+n-1}, h_1(u_{t+n-1}))) - h_1(Au_{t+n-1} + F(u_{t+n-1}, h(u_{t+n-1}))) +$$
$$+ h_1(Au_{t+n-1} + F(u_{t+n-1}, h(u_{t+n-1}))) - h(Au_{t+n-1} + F(u_{t+n-1}, h(u_{t+n-1})))]$$

Taking the norms and using the triangle inequality, gives

$$\left| h_1(u_{t+n-1}) - h(u_{t+n-1}) \right| \leq \left\| B^{-1} \right\| \left\| G' \right\|_{X_{r_u,1,r_v,1}} \left| h_1(u_{t+n-1}) - h(u_{t+n-1}) \right| + \left\| B^{-1} \right\| \left\| h_1' \right\| \left\| G' \right\|_{X_{r_u,1,r_v,1}} \left| h_1(u_{t+n-1}) - h(u_{t+n-1}) \right| +$$
$$+ \left\| B^{-1} \right\| \left\| h_0(Au_{t+n-1} + F(u_{t+n-1}, h(u_{t+n-1}))) - h(Au_{t+n-1} + F(u_{t+n-1}, h(u_{t+n-1}))) \right\|$$

Because of the properties of the mappings G and h_1, the domain $X_{r_u,1,r_v,1}$ can be chosen in such a way that

$$1 - \left\| B^{-1} \right\| \left\| G' \right\|_{X_{r_u,1,r_v,1}} - \left\| B^{-1} \right\| \left\| G' \right\|_{X_{r_u,1,r_v,1}} \left\| h_1' \right\| > 0 ;$$

therefore,

$$\left| h_2(u_{t+n-1}) - h(u_{t+n-1}) \right| \leq \left(1 - \left\| B^{-1} \right\| \cdot \left\| G' \right\|_{X_{r_u,0,r_v,0}} - \left\| B^{-1} \right\| \cdot \left\| G' \right\|_{X_{r_u,0,r_v,0}} \cdot \left\| h_1' \right\|_{U_{r_u,1}} \right)^{-1} \left\| B^{-1} \right\| \cdot \left| h_1(u_{t+n}) - h(u_{t+n}) \right|$$

From this inequality, it follows that the distance between the approximation h_2 and the true solution h at the point u_{t+n-1} has the same order as the distance between h_1 and h, but at the point u_{t+n}, which is generally nearer to the origin than u_{t+n-1} owing to the stability property.

In other words, the mapping h_2 has the same order of approximation as h_1 but on a larger domain of definition.

Continuing such a recurrent procedure further, yields a sequence of the approximate policy functions

$$h_i(u) = -B^{-1}G(u, h_i(u)) - B^{-1}h_{i-1}(Au + F(u, h_i(u))) , \; i=1, 2,...,$$

on increasing domains, i.e. $U_{r_u,i-1} \subset U_{r_u,i}$, and with any pre-defined accuracy. The algorithm can be continued until the inequalities

$$\left(1 - \left\| B^{-1} \right\| \left\| G' \right\|_{X_{r_u,i,r_v,i}} - \left\| B^{-1} \right\| \left\| G' \right\|_{X_{r_u,i,r_v,i}} \left\| h_i' \right\| \right) > 0 \qquad \text{and}$$

$$\left| Au_{t+n-i} + F(u_{t+n-i}, h(u_{t+n-i})) \right| < \left| u_{t+n-i} \right|, \; i = 1,2,..., \text{ hold. Appendix B provides the}$$
proof of the existence and convergence to the true solution of these approximate policy functions.

2.6 Discussion

Having the approximate mapping h_i, one can reconstruct the policy function (6) – (7) expressed in terms of the original by using the transformation Z from (12)

$$\begin{pmatrix} z_t \\ x_t \\ y_t \end{pmatrix} = Z \begin{pmatrix} u_t \\ h_i(u_t) \end{pmatrix} + \begin{pmatrix} 0 \\ \bar{x} \\ \bar{y} \end{pmatrix}.$$

The policy function $y(u_t) = h_i(x(u_t), z(u_t))$ is determined as a parametric function with parameter u_t

$$y_t = Z_3 \begin{pmatrix} u_t \\ h_i(u_t) \end{pmatrix} + \bar{y},$$

and the dynamics of the state variable is given by

$$x_{t+1} = Z_2 \begin{pmatrix} Au_t + F(u_t, h_i(u_t)) \\ h_i(Au_t + F(u_t, h_i(u_t))) \end{pmatrix} + \bar{x}$$

where Z_2 is a block of the block-decomposition of the matrix Z:

$$Z = \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix}.$$

Lipton et al. (1982) propose the multiple shooting method to solve non-linear rational expectations models. The shooting method is computationally unstable due to the saddle point property of these models. Incorrect guesses of initial values of the algorithm lead to very large errors in terminal values. By contrast, in the presented method the instability, which is determined by the norm of the operator B^{-1}, plays a positive role since it accelerates the convergence of the computational process and extends the domain of definition for solutions.

The constant L in (17) is an upper bound for the Lipschitz constants of the mappings G and F; hence, in all estimations above, the terms $\left\| G^{'} \right\|_{X_{r_{u,0}, r_{v,0}}}$ and $\left\| F^{'} \right\|_{X_{r_{u,0}, r_{v,0}}}$ can be replaced by L. Because of this, out of a steady state one can impose the Lipschitz continuity condition on these mappings instead of differentiability. This means that the proposed approach for constructing policy functions does not require the mappings involved to be differentiable; therefore, the method can handle non-differentiable problems such as the zero lower bound.

To sum up, the algorithm of the proposed method has the following steps:
1) Find a steady state;
2) Linearise the system (3) around the steady state;
3) Transform the system by introducing new variables u and v corresponding to the stable and unstable subspaces respectively;
4) Construct a stable manifold around the steady state for the transformed system using the contraction mapping iterations;
5) Return to the initial variables x, y and z by the transformation Z.

The algorithm is easy to implement as steps 1, 2, 3 and 5 can be implemented by using the existing software, for example, Dynare. The implementation of step 4 is simple because it invokes only the values of the mappings G and F at different values of arguments. Using Newton's method instead of mapping iterations may significantly accelerate the convergence of the computation process. However, Newton's method imposes higher requirement on the degree of smoothness for mappings; furthermore, the domain, where the algorithm converges, may be smaller.

2.7 Connection with the Extended Path Method

There is an interesting connection between the mappings h_i, $i = 1, 2, 3, ...$, and the extended path method proposed by Fair and Taylor (1983). Indeed, let us assume that variable x_t is exogenous, i.e. $F(x, y) = F(x)$. Then, the extended-path method applied to the transformed system (14) involves the following steps:

1. Fix a horizon n, and the terminal value $y_{t+n+1} = 0$;

2. Make a guess $Y_{n,t+i}^0 = 0$, $(i = 1,...,n)$;

3. If $Y_{n,t+i}^j$ is the approximation for y_{t+1} in iteration j, then the next iterate $Y_{n,t+i}^{j+1}$ is implicitly defined by Type I iteration (according to Fair and Taylor notation).

Hence

$$Y_{n,t+i}^{j+1} = -B^{-1}G(x_{t+i}, Y_{n,t+i}^{j+1}) + B^{-1}Y_{n,t+i+1}^{j+1} \text{ ,, } i = 0,..., n;$$

4. Repeat step 4 for j, $j = 1,..., T$. These iterations are called Type II iterations.

The first iteration of Type II, i.e. $j = 1$, gives the approximation
$$Y_{n,t+i}^1 = h_1(x_{t+i}), i = 0,...,n.$$

Therefore, the value $Y_{n,t+i}^1$ equals the value of the mapping h_1 at the point x_{t+i}. Completing n iterations of Type II, results in
$$Y_{n,t}^n = h_n(x_t), \ Y_{n,t+1}^n = h_{n-1}(x_{t+1}),..., \ Y_{n,t+i}^n = h_{n-i}(x_{t+i}),..., \ Y_{n,t+n}^n = h_0(x_{t+n}) = 0.$$

This means that after Type II iteration the extended path method gives the solution values at the time $t+i$ equal to the values of the mappings h_i at the points x_{t+i} respectively, $i=1,2,..., n$. In other words, the value of the mapping h_i at the point x_{t+i} corresponds to the Fair–Taylor solution at $t+i$.

Gagnon and Taylor (1990) mention that there is no proof that the extended path method converges to the true rational expectation solution for non-linear models. The presented approach can be employed straightforwardly for general and rigorous proof of the convergence of the extended path method (see Appendix B). Moreover,

the proposed method has additional advantages over the Fair–Taylor approach. First, the stable manifold method has a wider domain of convergence than the conventional extended path method. Indeed, if the Jacobians at the steady state for the mappings F and G are non-zero[5], then in a general case the Lipschitz constant of the operator involved in the contraction mapping theorem (see Appendix B for details) reaches the value of 1 in a smaller ball U_r than in the proposed approach in which the Jacobians are zero at the steady state owing to the transformation of coordinate (12). If the Lipschitz constant of the operator is greater than or equal to 1, then this operator becomes non-contracting, and hence a solution to the problem might not exist.

Second, the algorithm converges faster than the extended path method for a similar reason; namely, the Lipschitz constant on a definite domain is smaller for the proposed approach than for the Fair-Taylor method.[6]

3. THE STOCHASTIC CASE

In this Section, we shall briefly outline how the results obtained in the previous Section can be extended to the stochastic case. For the sake of simplicity, we omit the vector of the state variables x_t of the endogenous state variables and leave only exogenous state variables z_t in the model (4) – (5). Specifically, consider the system

$$E_t f(y_{t+1}, y_t, z_t, \varepsilon_{t+1}) = 0 \quad (26),$$

$$z_{t+1} = \Lambda z_t + \sigma \varepsilon_{t+1}, \ \varepsilon_t \sim N(0, \ \Omega) \ (27).$$

In much the same way and under the same conditions as in Subsection 2.3, we can transform the system (26) – (27) by introducing new variables u_t and v_t corresponding to the stable and unstable subspaces respectively. The transformed system has the following form:

$$u_{t+1} = \Lambda u_t + \sigma \varepsilon_{t+1},$$
$$E_t v_{t+1} = B v_t + E_t G(u_t, v_t, v_{t+1}, \sigma \varepsilon_{t+1})$$

where the mapping G is continuously differentiable and vanishes together with its first derivatives at $(0,0,0,0)$. Note that unlike the deterministic case it is hardly possible here to obtain the representation (14) where the mapping G does not depend on v_{t+1}. For example, the cross-term with the innovations $E_t(v_{t+1}\varepsilon_{t+1})$ cannot be split.

By the same argument as in the previous Section, it can be shown that there exists a map h_1 of $U_{r_{u,1}}$ to $V_{r_{v,1}}$ such that

$$h_1(u_t, \sigma) = -B^{-1} E_t G(u_t, h_1(u_t, \sigma), 0, \sigma \varepsilon_{t+1}).$$

---

[5] This is the case for the Fair-Taylor method applied to general rational expectations models.
[6] From the contraction mapping theorem, it follows easily that the smaller the Lipschitz constant, the faster the convergence to the fixed point.

Furthermore, there exists a sequence of maps $h_i : U_{r_{u,i}} \to V_{r_{v,i}}$, $i = 2, 3, ...,$

$$h_{i+1}(u_t, \sigma) = -B^{-1} E_t G(u_t, h_{i+1}(u_t), h_i(\Lambda u_t + \sigma \varepsilon_{t+1}), \sigma \varepsilon_{t+1}) + B^{-1} E_t h_i(\Lambda u_t + \sigma \varepsilon_{t+1})$$

where $U_{r_{u,i}}$ and $V_{r_{v,i}}$ are some balls in R^{n_z} and R^{n_y} respectively.

Similarly to Jin and Judd (2002), under a certain regularity condition using the implicit function theorem it can be proved that the mappings $h_{i,\sigma}$, $i \in N$, continuously depend on the parameter σ; moreover, $h_{i,\sigma}$ tend to h_i in C^0-topology as $\sigma \to 0$, where h_i are the mappings corresponding to the deterministic case.

Note that from a practical point of view to obtain the mappings h_i, $i = 1, 2,...$ in the stochastic case we cannot implement iterative methods directly in step 4 of the algorithm mentioned in Subsection 2.4. Nevertheless, the analogy with the extended path method would suggest that the approach proposed by Adjemian and Juillard (2010) and called the stochastic extended path approach can also be applied in our case. This approach implies that the conditional expectations are computed employing either quadratures or some stochastic simulation algorithms. Another possible approach is to expand the policy function h in powers of σ.

4. EXAMPLE: THE NEOCLASSICAL GROWTH MODEL

This Section applies the method presented above to the neoclassical growth model (the Brock–Mirman (1972) model). Consider the deterministic one-sector growth model with inelastic labour supply. The representative agent maximises the inter-temporal utility function

$$\max_{\{c_t, k_t\}} \sum_{t=0}^{\infty} E_0 \{\beta^t \ln(c_t)\}$$

subject to

$$c_t + k_{t+1} = k_t^{\alpha} .$$

Using the resource constraint to substitute out consumption, we have the following equilibrium condition:

$$\frac{1}{k_t^{\alpha} - k_{t+1}} = \beta \frac{\alpha}{(k_{t+1}^{\alpha} - k_{t+2}) k_{t+1}^{1-\alpha}} .$$

This model has an analytical solution for the policy function that is given by

$$k_{t+1} = \tilde{h}(k_t) = \alpha \beta k_t^{\alpha} .$$

We calculate approximations in the level (rather than in the logarithm) of the state variable, otherwise the problem becomes trivially linear. The parameter values take on standard values, namely: α = 0.36, β = 0.99. Then for our calibration the steady state value of capital is $\bar{k} = (\alpha\beta)^{\frac{1}{1-\alpha}} = 0.20$. It is not hard to see that the Taylor series expansion of the true solution (35) converges in the interval (0, $2\bar{k}$).

Let us now check the accuracy of the methods presented above. Since the model has a closed-form solution, we can check the accuracy of each approximate solution against the true one. As our main focus is on whether an approximation is globally valid, we compare the accuracy of different solutions at the points that are situated near or at the endpoints of the Taylor series convergence interval: namely, $k = 0.05$ and $k = 2\bar{k} \pm 0.05$, and $k = 2\bar{k}$ respectively. One point ($k = 0.9$) is chosen far away from the Taylor series convergence interval and, consequently, from the steady state.

Table 1 reports the relative errors measured in percentage points for the functions h_1, h_2 and h_3 constructed by the presented method and the Taylor series expansion of the 1st, 2nd, 5th and 16th order. We also consider the first iteration in approximating the function h_1 and denote this function by $h_{1,1}$. The function $h_{1,1}$ can be obtained by inserting $v=0$ into (20): $h_{1,1} = -B^{-1}G(u,0)$. The explicit form of this function is derived in Appendix A.

Table 1 shows that the function h_3 has the best accuracy for all points under consideration. Even for points lying within the domain of the Taylor series convergence it is more accurate than the 16th-order Taylor series expansion. The function h_2 has also very high accuracy with the maximum relative error of only 0.1% at the point $k = 0.05$. The function h_1 is more accurate than the 5th-order Taylor series expansion at the points within the Taylor series expansion convergence domain. The function $h_{1,1}$, which is the simplest solution among the solutions under study (except for the linear one), gives a relatively good approximation at $k = 0.05$, $2\bar{k}$, $2\bar{k} - 0.05$ and $2\bar{k} + 0.05$. Its approximation accuracy is comparable to the 5th-order Taylor series expansion and considerably higher than the 2nd-order Taylor series expansion at the points within the Taylor series convergence interval. At the point $k = 0.9$, which is far away from the steady state, the functions h_1, h_2, and h_3 give very good accuracy: the approximation error is 1%, 0.03%, and 0.003% respectively. By contrast, the Taylor series expansions are extremely bad at this point.

Figure 2 complements the results. Different solutions of the capital policy function are plotted restricting the argument k_t in the interval [0, $5\bar{k}$]. All approximations are quite close to the true solution in the neighbourhood of the steady state of radius $0.25\bar{k} = 0.05$ The 1st- and 2nd-order Taylor series approximations start to diverge from the true solution outside the interval [0.1, 0.3]. The 5th- and 16th-order Taylor series approximations perform well in the interval [0, 0.40], i.e. within the domain of the convergence of the Taylor series expansion; however, outside the interval they explode. The function h_1 provides a very close fit for the whole interval. The function $h_{1,1}$ is fairly accurate in the interval [0.05, 0.5]. The solution h_2 is essentially indistinguishable from the true solution for all k_{t-1}, thus providing the perfect global approximation (for this reason, the graph of h_3, which is also indistinguishable from the graph of h, is omitted). Another feature of the functions $h_{1,1}$, h_1 and h_2 noticeably distinguishing them from the Taylor series expansions is that they preserve the form of the true solution, i.e. they are monotonically increasing and concave. This feature is not surprising. By construction, the functions $h_{1,1}$, h_1 and h_2 contain implicitly all information about the global behaviour of the true solution h, whereas the Taylor series contain only information about the local behaviour of h.

Table 1. The relative errors of the approximate solutions at different points

Model	0.05	$2\bar{k}-0.05$	$2\bar{k}$	$2\bar{k}+0.05$	0.9
ASM					
$h_{1,1}$	0.81	0.62	1.26	2.03	15.14
h_1	0.94	0.17	0.28	0.37	0.96
h_2	0.10	0.01	0.03	0.02	0.03
h_3	0.02	0.001	0.01	0.004	0.003
The orders of Taylor series expansion					
1st	20.2	3.64	4.73	8.26	31.45
2nd	9.44	−1.40	−3.01	−5.44	−51.90
5th	1.67	0.20	1.02	3.35	711.98
16th	0.02	−0.002	−0.20	−8.59	−1.65E+06

Notes: 1) ASM – approximate stable manifolds. 2) The relative errors are measured in percentage points.

Figure 2. Policy functions for different approximations
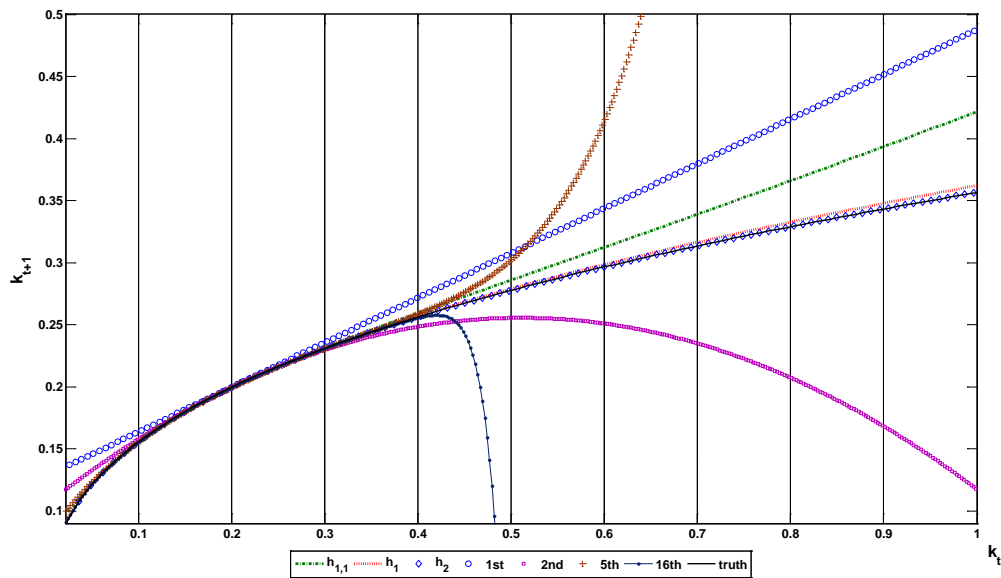
CONCLUSION

The present paper proposes a new method to construct the solutions to non-linear rational expectations models on non-local domains. The method involves using the successive approximations procedure which invokes only the values of the non-linear terms. The approach allows us to derive the estimate of the accuracy and domain of the approximate solutions. By construction, the proposed solutions are non-local and cannot explode. This distinguishes the approach from the perturbation methods. In contrast to the projection methods, the presented algorithm finds the policy function only at a number of the forward solution points rather than the whole domain of definition. This feature implies less time-consuming computations compared with those of projection methods. Such an approach is preferable to compute a particular perfect foresight solution, for example, a solution with specific initial conditions (a specific impulse response function).

The application of the method to the neoclassical growth model shows that the proposed solutions are as accurate as the high-order Taylor series expansions locally and definitely much better globally, i.e. far away from the steady state. In addition, the solutions inherit the global properties of the true solution such as monotonicity and concavity. The simplicity of the implementation of the method allows for incorporating the algorithm into existing software platforms such as Dynare; therefore, they can be applied to more complex models.

Since the mappings involved in the algorithm approach require a less restrictive condition than differentiability, i.e. the Lipschitz continuity, the method can, in principle, deal with the zero lower bound problem. Theoretically, the method may be extended to the stochastic case in a fairly straightforward way. The author leaves the practical implementation of the approach to stochastic models for future research.

APPENDICES

Appendix A. The Neoclassical Growth Model

The representative agent maximises

$$\max_{\{c_t, k_t\}} \sum_{t=0}^{\infty} E_0 \{\beta^t \ln(c_t)\} \quad \text{(A1)}$$

subject to

$$c_t + k_{t+1} = k_t^{\alpha} \quad \text{(A2)}.$$

Using the resource constraint to substitute out consumption, we have the following equilibrium condition:

$$\frac{1}{k_t^{\alpha} - k_{t+1}} = \beta \frac{\alpha}{(k_{t+1}^{\alpha} - k_{t+2})k_{t+1}^{1-\alpha}} \quad \text{(A3)}.$$

The true solution of the policy function is given by

$$k_{t+1} = \alpha\beta k_t^{\alpha} \quad \text{(A4)}.$$

Inverting (A3), gives

$$k_t^{\alpha} - k_{t+1} = \frac{1}{\beta\alpha}(k_{t+1}^{\alpha} - k_{t+2})k_{t+1}^{1-\alpha} .$$

Expressing k_{t+2} as a function of k_{t+1} and k_t, yields

$$k_{t+2} = \frac{(1+\alpha\beta)k_{t+1} - \alpha\beta k_t^{\alpha}}{k_{t+1}^{1-\alpha}} \quad \text{(A5)}.$$

Taking into account the steady state

$$\bar{k} = (\alpha\beta)^{\frac{1}{1-\alpha}} \quad \text{(A6)},$$

we get

$$\hat{k}_{t+2} = (1+\alpha\beta)(\bar{k} + \hat{k}_{t+1})^{\alpha} - \frac{\alpha\beta(\bar{k} + \hat{k}_t)^{\alpha}}{(\bar{k} + \hat{k}_{t+1})^{1-\alpha}} - \bar{k} \quad \text{(A7)}.$$

Equation (7A) can be presented as

$$\hat{k}_{t+2} = (1+\alpha\beta)\bar{k}^{\alpha}(1+\frac{\hat{k}_{t+1}}{\bar{k}})^{\alpha} - \frac{\alpha\beta\bar{k}^{\alpha}(1+\frac{\hat{k}_t}{\bar{k}})^{\alpha}}{\bar{k}^{1-\alpha}(1+\frac{\hat{k}_{t+1}}{\bar{k}})^{1-\alpha}} - \bar{k} = f(\hat{k}_{t+1}, \hat{k}_t) \quad \text{(A8)}.$$

Now we extract the linear part of the right-hand side of (A8):

$$f(\hat{k}_{t+1},\hat{k}_t) = (1+\alpha\beta)\bar{k}^\alpha (1+\alpha\frac{\hat{k}_{t+1}}{\bar{k}} + o(\hat{k}_{t+1})) - \frac{\alpha\beta\bar{k}^\alpha(1+\alpha\frac{\hat{k}_t}{\bar{k}} + o(\hat{k}_t))}{\bar{k}^{1-\alpha}(1+(1-\alpha)\frac{\hat{k}_{t+1}}{\bar{k}} + o(\hat{k}_{t+1}))} - \bar{k}$$

$$= (1+\alpha\beta)\bar{k}^\alpha + (1+\alpha\beta)\bar{k}^\alpha\alpha\frac{\hat{k}_{t+1}}{\bar{k}} - \frac{\alpha\beta\bar{k}^\alpha}{\bar{k}^{1-\alpha}}(1+\alpha\frac{\hat{k}_t}{\bar{k}})[1-(1-\alpha)\frac{\hat{k}_{t+1}}{\bar{k}}] + o(\hat{k}_t,\hat{k}_{t+1,}) - \bar{k}$$

$$= (1+\alpha\beta)\bar{k}^\alpha - \frac{\alpha\beta\bar{k}^\alpha}{\bar{k}^{1-\alpha}} + (1+\alpha\beta)\bar{k}^\alpha\alpha\frac{\hat{k}_{t+1}}{\bar{k}} - \frac{\alpha^2\beta}{\bar{k}^{2(1-\alpha)}}\hat{k}_t + (1-\alpha)\frac{\hat{k}_{t+1}}{\bar{k}^{2(1-\alpha)}} + o(\hat{k}_t,\hat{k}_{t+1,}) - \bar{k}$$

Taking into account that $\bar{k} = (1+\alpha\beta)\bar{k}^\alpha - \frac{\alpha\beta\bar{k}^\alpha}{\bar{k}^{1-\alpha}}$ and $\bar{k} = (\alpha\beta)^{\frac{1}{1-\alpha}}$, we obtain

$$f(\hat{k}_{t+1},\hat{k}_t) = (1+\alpha\beta)\alpha\frac{\hat{k}_{t+1}}{\bar{k}^{1-\alpha}} - \frac{\alpha^2\beta}{\bar{k}^{2(1-\alpha)}}\hat{k}_t + (1-\alpha)\alpha\beta\frac{\hat{k}_{t+1}}{\bar{k}^{2(1-\alpha)}} + o(\hat{k}_t,\hat{k}_{t+1}) =$$

$$= (1+\alpha\beta)\alpha\frac{\hat{k}_{t+1}}{\alpha\beta} - \frac{\alpha^2\beta}{(\alpha\beta)^2}\hat{k}_t + (1-\alpha)\alpha\beta\frac{\hat{k}_{t+1}}{(\alpha\beta)^2} + o(\hat{k}_t,\hat{k}_{t+1}) = (1+\alpha\beta)\frac{\hat{k}_{t+1}}{\beta} - \frac{1}{\beta}\hat{k}_t + (1-\alpha)\frac{\hat{k}_{t+1}}{\alpha\beta} + o(\hat{k}_t,\hat{k}_{t+1}) =$$

$$= (\frac{1}{\beta}+\alpha+\frac{1}{\alpha\beta}-\frac{1}{\beta})\hat{k}_{t+1} - \frac{1}{\beta}\hat{k}_t + o(\hat{k}_t,\hat{k}_{t+1}) = (\frac{1}{\alpha\beta}+\alpha)\hat{k}_{t+1} - \frac{1}{\beta}\hat{k}_t + o(\hat{k}_t,\hat{k}_{t+1})$$

therefore, equation (A8) has the following form:

$$\hat{k}_{t+2} = -\frac{1}{\beta}\hat{k}_t + (\frac{1}{\alpha\beta}+\alpha)\hat{k}_{t+1} + \left[(1+\alpha\beta)(\bar{k}+\hat{k}_t)^\alpha - \frac{\alpha\beta(\bar{k}+\hat{k}_t)^\alpha}{(\bar{k}+\hat{k}_{t+1})^{1-\alpha}} - \bar{k} + \frac{1}{\beta}\hat{k}_t - (\frac{1}{\alpha\beta}+\alpha)\hat{k}_{t+1}\right] (A9).$$

If we denote $\hat{k}_{t+1}$ by $\hat{z}_t$, then equation (A9) can be rewritten as the following system:

$$\hat{k}_{t+1} = \hat{z}_t,$$
$$\hat{z}_{t+1} = -\frac{1}{\beta}\hat{k}_t + (\frac{1}{\alpha\beta}+\alpha)\hat{z}_t + N_1(\hat{k}_t,\hat{z}_t) \quad (A10)$$

where the non-linear term is

$$N_1(\hat{k}_t,\hat{z}_t) = (1+\alpha\beta)(\bar{k}+\hat{z}_t)^\alpha - \frac{\alpha\beta(\bar{k}+\hat{k}_t)^\alpha}{(\bar{k}+\hat{z}_t)^{1-\alpha}} - \bar{k} - (\frac{1}{\alpha\beta}+\alpha)\hat{z}_t + \frac{1}{\beta}\hat{k}_t.$$

Rewriting (A10) in a matrix form, gives

$$\begin{bmatrix} \hat{k}_{t+1} \\ \hat{z}_{t+1} \end{bmatrix} = L\begin{bmatrix} \hat{k}_t \\ \hat{z}_t \end{bmatrix} + \begin{bmatrix} 0 \\ N_1(\hat{k}_t,\hat{z}_t) \end{bmatrix}, (A11)$$

where matrix L has the following form:

$$L = \begin{bmatrix} 0 & 1 \\ -\dfrac{1}{\beta} & (\dfrac{1}{\alpha\beta} + \alpha) \end{bmatrix}.$$

Next L is transformed into the Jordan canonical form:

$L = ZJZ^{-1}$ where

$$J = \begin{pmatrix} \alpha & 0 \\ 0 & \dfrac{1}{\alpha\beta} \end{pmatrix}, \qquad\qquad Z = \begin{pmatrix} 1 & 1 \\ \alpha & \alpha\beta \end{pmatrix},$$

$$Z^{-1} = \begin{pmatrix} \dfrac{1}{\alpha\beta} & -1 \\ -\alpha & 1 \end{pmatrix} \dfrac{\alpha\beta}{1-\alpha^2\beta} = \begin{pmatrix} \dfrac{1}{1-\alpha^2\beta} & -\dfrac{\alpha\beta}{1-\alpha^2\beta} \\ -\dfrac{1}{\dfrac{1}{\alpha^2\beta}-1} & \dfrac{\alpha\beta}{1-\alpha^2\beta} \end{pmatrix}.$$

After introducing new variables $\begin{bmatrix} u_t \\ v_t \end{bmatrix} = Z^{-1}\begin{bmatrix} \hat{k}_t \\ \hat{z}_t \end{bmatrix}$, and pre-multiplying (11A) by Z^{-1},

we can rewrite (5A) as

$$\begin{aligned} u_{t+1} &= Au_t + F(u_t, v_t) \\ v_{t+1} &= Bv_t + G(u_t, v_t) \end{aligned} \quad \text{(A12)}$$

where

$$A = \alpha, \; B = 1/(\alpha\beta), \; F(u_t, v_t) = Z^{12}N_1(Z_{11}u_t + Z_{12}v_t, Z_{21}u_t + Z_{22}v_t),$$
$$G(x_t, y_t) = Z^{22}N_1(Z_{11}x_t + Z_{12}y_t, Z_{21}x_t + Z_{22}y_t) \quad \text{(A13)}$$

where Z_{ij} and Z^{ij} are components of matrixes Z and Z^{-1} respectively.

In the explicit form, the system (A12) can be written as

$$u_{t+1} = \alpha u_t - \frac{\alpha\beta}{1-\alpha^2\beta}\left[(1+\alpha\beta)(\bar{k}+\alpha u_t + \frac{1}{\alpha\beta}v_t)^\alpha - \frac{\alpha\beta(\bar{k}+u_t+y_t)^\alpha}{(\bar{k}+\alpha u_t + \frac{1}{\alpha\beta}v_t)^{1-\alpha}} - \bar{k} - (\frac{1}{\alpha\beta}+\alpha)(\alpha u_t + \frac{1}{\alpha\beta}v_t) + \frac{1}{\beta}(u_t+v_t) \right]$$

$$v_{t+1} = \frac{1}{\alpha\beta}v_t + \frac{\alpha\beta}{1-\alpha^2\beta}\left[(1+\alpha\beta)(\bar{k}+\alpha u_t + \frac{1}{\alpha\beta}v_t)^\alpha - \frac{\alpha\beta(\bar{k}+u_t+y_t)^\alpha}{(\bar{k}+\alpha u_t + \frac{1}{\alpha\beta}v_t)^{1-\alpha}} - \bar{k} - (\frac{1}{\alpha\beta}+\alpha)(\alpha u_t + \frac{1}{\alpha\beta}v_t) + \frac{1}{\beta}(u_t+v_t) \right].$$

Then the function $h_1(u_t)$ has the following form:

$$h_1(u_t) = -\frac{(\alpha\beta)^2}{1-\alpha^2\beta}\left[(1+\alpha\beta)(\bar{k}+\alpha u_t + \frac{1}{\alpha\beta}h_1(u_t))^\alpha - \frac{\alpha\beta(\bar{k}+u_t+h_1(u_t))^\alpha}{(\bar{k}+\alpha u_t + \frac{1}{\alpha\beta}h_1(u_t))^{1-\alpha}} - \bar{k} - (\frac{1}{\alpha\beta}+\alpha)(\alpha u_t + \frac{1}{\alpha\beta}h_1(u_t)) + \frac{1}{\beta}(u_t+h_1(u_t)) \right] \quad \text{(A14)}.$$

The function $h_{1,1}$ is obtained by substituting zeros for $h_1(u_t)$ in the right-hand side of (A14):

$$h_{11}(u_t) = -\frac{(\alpha\beta)^2}{1-\alpha^2\beta}\left[(1+\alpha\beta)(\bar{k}+\alpha u_t)^\alpha - \frac{\alpha\beta(\bar{k}+u_t)^\alpha}{(\bar{k}+\alpha u_t)^{1-\alpha}} - \bar{k} - \alpha^2 u_t\right].$$

Having the mapping $h_i(u_t)$ to return to the original variables, one must perform transformation Z:

$$\begin{bmatrix} \hat{k}_t \\ \hat{k}_{t+1} \end{bmatrix} = Z\begin{bmatrix} u_t \\ h_i(u_t) \end{bmatrix}.$$

Then the policy function $\tilde{h}(\hat{k}_t)$ has the following parametric representation:

$$\begin{cases} \hat{k}_t = u_t + h_i(u_t); \\ \tilde{h}(\hat{k}_t) = \alpha u_t + \dfrac{1}{\alpha\beta}h_i(u_t) \end{cases} \text{(A15).}$$

Appendix B

The present theorem proves the existence of a sequence of approximate stable manifolds.

Theorem

Let X_{r_u,r_v} be the domain of definition for the mappings F and G in (14) such that the following conditions hold:

(i) $$\frac{\left\|B^{-1}\right\|\left\|G\right\|_{X_{r_u,r_v}}}{1-\left\|B^{-1}\right\|} < r_v \; ;$$

(ii) $$\frac{1-\left\|B^{-1}\right\|\left\|A\right\|}{4\left\|B^{-1}\right\|} > L$$

where

$$L = \max(\left\|G^{'}\right\|_{X_{r_u,r_v}}, \left\|F^{'}\right\|_{X_{r_u,r_v}}) \; \text{(B1)};$$

(iii) the stability condition: if $(u_t, v_t) \in X_{r_u,r_v}$, then $u_{t+1} = Au_t + F(u_t, v_t) \in U_{r_u}$.

Then there exists a sequence of the mappings $h_i : U_{r_u} \to V_{r_v}$, i = 0, 1, 2, ..., satisfying the recurrent equations:

$$h_i(u) = -B^{-1}G(u, h_i(u)) + B^{-1}h_{i-1}(Au + F(u, h_i(u))) \; \text{(B2)}$$

with the initial condition $h_0 \equiv 0$. Moreover, the following inequalities for the norm of the mappings h_i:

a. $$\left\|h_i\right\|_{U_r} \leq \left(\frac{1-\left\|B^{-1}\right\|^{i+1}}{1-\left\|B^{-1}\right\|}\right)\left\|B^{-1}\right\| \cdot \left\|G\right\|_{X_{r_u,r_v}} \leq r_u \; \text{(B3)};$$

b. $$\left\|h_i^{'}\right\|_{U_r} \leq \frac{1-\left\|B^{-1}\right\|L}{\left\|B^{-1}\right\|L} \; \text{(B4)}$$

hold.

Remark

The neighbourhood X_{r_u,r_v} that satisfies conditions 1, 2 and 3 always exists locally, because the mappings $G(u,v)$ and $F(u,v)$ vanish, together their first derivatives, at *(0,0)*. Nonetheless, these conditions are not local by themselves.

Proof

The proof is by induction on *i*. More precisely, using the contraction mapping theorem, we will derive by induction on $i \in N$ the existence of h_i satisfying (B2). To satisfy the conditions of the contraction mapping theorem, we need the estimates

(B3) and (B4) for each stage of the induction. Suppose that $i = 1$. Let $T_{1,u}$ be the parameterised mapping of V_{r_v} to R^{n_y} such that

$$T_{1,u}(v) = -B^{-1}G(u,v) \text{ (B5)}$$

for each $u \in U_{r_u}$. Suppose that $T_{1,u}$ satisfies the conditions of the contraction mapping theorem (Ljungqvist and Sargent (2004)) for every parameter u. Then there exists a fixed point h_1 of $T_{1,u}$ such that

$$h_1(u) = -B^{-1}G(u,h_1(u)) \, , \; u \in U_{r_u} \text{ (B6)}.$$

We claim that $T_{1,u}$ maps the closed ball V_{r_v} into itself and it has the Lipschitz constant less than one and thus satisfies the conditions of the contraction mapping theorem. Note also that the dependence of h_1 on u determines the mapping of U_{r_u} to R^{n_y}.

If h_1 satisfies the inequalities (B3) and (B4), then the induction hypothesis will be proved for $i=1$. Indeed, taking the norm of both sides (B5) and using condition 1, we have

$$\left\| T_{1,u} \right\|_{V_{r_v}} = \left\| B^{-1} \right\| \cdot \left\| G \right\|_{X_{r_u,r_v}} \leq r_v \text{ (B7)}.$$

This means that $T_{1,u}$ maps V_{r_v} into itself. Now our task is to show that $J_{1,u}$ is a contraction, i.e. that $T_{1,u}$ has the Lipschitz constant less than one. The Jacobian of $T_{1,u}$ is

$$T_{1,u}^{'}(v) = -B^{-1}G_v^{'}(u,v) \text{ (B8)}$$

where $G_v^{'}(u,v)$ is the Jacobian of the mapping G with respect to v at the point (u, v). Taking the norm of both sides (B8) and using (B1) and condition 2, we obtain

$$\left\| T_{1,u}^{'}(v) \right\| \leq \left\| B^{-1} \right\| L < 1 \text{ for all } (u,v) \in X_{r_u,r_v} \text{ (B9)}.$$

The norm $\left\| T_{1,u}^{'}(v) \right\|$ is the upper bound for the Lipschitz constant of $T_{1,u}$ in the domain V_{r_v}. Since the mapping $T_{1,u}$ has the Lipchitz constant less than one and maps the closed ball V_{r_v} into itself, we see that according to the contracting mapping theorem, the operator $T_{1,u}$ has a fixed point h_1 in V_{r_v} for each $u \in U_{r_u}$. This implies that the mapping h_1 defined by (B6) exists. From (B7) it follows that $\left\| h_1 \right\|_{U_{r_u}} \leq r_v$; hence the mapping h_1 satisfies Condition 3 of the theorem. It remains to check that the norm of the derivative of h_1 satisfies the inequality (B4).

Differentiating (B6) with respect to u gives

$$h_1(u)^{'} = B^{-1}G_u^{'}(u,h_1(u)) + B^{-1}G_v^{'}(u,h_1(u))h_1^{'}(u) \, .$$

Taking the norms and applying the triangle inequality, yields

$$\left\| h_1(u)^{'} \right\| \leq \left\| B^{-1} \right\| \cdot \left\| G_u^{'}(u, h_1(u)) \right\| + \left\| B^{-1} \right\| \cdot \left\| G_v^{'}(u, h_0(u)) \right\| \cdot \left\| h_1(u)^{'} \right\|, \; u \in U_{r_u} \; \text{(B10)}.$$

Rearranging the terms in (B10) and taking into account (B1) and the definitions $\left\| h_1^{'} \right\|_{U_{r_u}}$ and $\left\| G_u^{'} \right\|_{X_{r_u, r_v}}$, leads to

$$\left\| h_1^{'} \right\|_{U_{r_u}} \leq (1 - \left\| B \right\|^{-1} L)^{-1} \cdot \left\| B \right\|^{-1} L.$$

From Condition 2, it follows easily that $\left\| B \right\|^{-1} L < 1/2$. This implies that

$$\left\| h_1^{'} \right\|_{U_{r_u}} < (1 - \left\| B \right\|^{-1} L) / \left\| B \right\|^{-1} L;$$

hence, h_1 satisfies the inequality (B4). Therefore the inductive assumption is proved for $i=1$.

Next, suppose inductively that there exist i mappings $h_k(u)$, $k = 1, 2, ..., i$ that satisfy Conditions 1–3. Let $T_{i+1,u}$ be the parameterised mapping of V_{r_v} to R^{n_y} such that

$$T_{i+1,u}(v \;) = -B^{-1} G(u, v) + B^{-1} h_i(Au + F(u, v)) \; \text{(B11)}$$

for each $u \in U_{r_u}$. As before, we shall show that $T_{i+1,u}$ satisfies the contraction mapping theorem conditions. Indeed, taking the norms in (B11) and applying the triangle inequality, yields

$$\left\| T_{i+1,u} \right\|_{V_{r_v}} \leq \left\| B^{-1} \right\| \left\| G \right\|_{X_{r_u, r_v}} + \left\| B^{-1} \right\| \left\| h_i \right\|_{U_{r_u}} \; \text{(B12)}.$$

By the inductive assumption, the inequality (B4) holds; therefore,

$$\left\| T_{i+1,u} \right\|_{V_{r_v}} \leq \left\| B^{-1} \right\| \left\| G \right\|_{X_{r_u, r_v}} + \left\| B^{-1} \right\| \left(\frac{1 - \left\| B \right\|^{-i-1}}{1 - \left\| B \right\|^{-1}} \right) \left\| B \right\|^{-1} \left\| G \right\|_{X_{r_u, r_v}} = \left\| B^{-1} \right\| \left(\frac{1 - \left\| B \right\|^{-i-2}}{1 - \left\| B \right\|^{-1}} \right) \left\| G \right\|_{X_{r_u, r_v}} \leq r_v$$

(B13)

where the last inequality follows from Condition 1. This means that $T_{1,u} : V_{r_v} \to V_{r_v}$ for all $u \in U_{r_u}$. The Jacobian of the mapping $T_{i+1,u}$ is

$$T_{i+1,u}^{'}(v) = -B^{-1} G_v^{'}(u, v) + B^{-1} h_i^{'}(Au + F(u, v)) F_v^{'}(u, v).$$

Taking the norms, using Condition 3 and applying (B1), gives

$$\left\| T_{i+1,u}^{'} \; (v) \right\| \leq \left\| B^{-1} \right\| L + \left\| B^{-1} \right\| \left\| h_i^{'} \right\|_{U_{r_u}} L \; \text{(B14)}$$

for all $v \in V_{r_v}$. By the inductive assumption, $\left\| h_i^{'} \right\|_{U_r} \leq \dfrac{1 - \left\| B^{-1} \right\| L}{\left\| B^{-1} \right\| L}$; hence,

$$\left\| T_{i+1,u_{t+n-i-1}}^{'} \right\| < 1.$$

Since the mapping $T_{i+1,u}$ (v) has the Lipchitz constant less than one and maps V_{r_v} into itself, we see that according to the contraction mapping theorem, the mapping $T_{i+1,u}$ has a fixed point h_{i+1} in V_{r_v} for each $u \in U_{r_u}$. This implies that there exists a mapping h_{i+1} of U_{r_u} to V_{r_v} such that

$$h_{i+1}(u) = -B^{-1}G(u, h_{i+1}(u)) - B^{-1}h_i(Au + F(u, h_{i+1}(u))) \ \text{(B15)}.$$

From (B12), it follows that

$$\|h_{i+1}\| \leq \left(\frac{1 - \|B\|^{-i-2}}{1 - \|B\|^{-1}} \right) \|B\|^{-1} \|G\|_{X_{r_u, r_v}}.$$

Hence the mapping h_{i+1} satisfies Condition 1. To conclude the inductive assumption for $i+1$, it remains to check the inequality (B5) for the norm of the derivative of the mapping h_{i+1}. Indeed, taking the derivative of h_{i+1} at the point u, yields

$$h_{i+1}'(u_{t+n-i-1}) = -B^{-1}G_x'(u_{t+n-i-1}, h_{i+1}(u_{t+n-i-1})) - B^{-1}G_y'(u_{t+n-i-1}, h_{i+1}(u_{t+n-i-1})) \cdot h_{i+1}'(u_{t+n-i-1}) -$$

$$- B^{-1}h_i'(Au_{t+n-i-1} + F(u_{t+n-i-1}, h_{i+1}(u_{t+n-i-1}))) \times$$

$$\times \left(A + F_u'(u_{t+n-i-1}, h_{i+1}(u_{t+n-i-1})) + F_v'(u_{t+n-i-1}, h_{i+1}(u_{t+n-i-1}))h_{i+1}'(u_{t+n-i-1}) \right)$$

Taking the norms and using the triangle inequality, gives

$$\left\| h_{i+1}'(u_{t+n-i-1}) \right\| \leq \left\| B^{-1} \right\| \left\| G_u'(u_{t+n-i-1}, h_{i+1}(u_{t+n-i-1})) \right\| + \left\| B^{-1} \right\| \left\| G_v'(u_{t+n-i-1}, h_{i+1}(u_{t+n-i-1})) \right\| \cdot \left\| h_{i+1}'(u_{t+n-i-1}) \right\| +$$

$$+ \left\| B^{-1} \right\| \left\| h_i'(Au_{t+n-i-1} + F(u_{t+n-i-1}, h_{i+1}(u_{t+n-i-1}))) \right\| \cdot \left\| A + F_u'(u_{t+n-i-1}, h_{i+1}(u_{t+n-i-1})) \right\| +$$

$$+ \left\| B^{-1} \right\| \left\| h_i'(Au_{t+n-i-1} + F(u_{t+n-i-1}, h_{i+1}(u_{t+n-i-1}))) \right\| \cdot \left\| F_v'(u_{t+n-i-1}, h_{i+1}(u_{t+n-i-1}))h_{i+1}'(u_{t+n-i-1}) \right\|$$

for all $u_{t+n-i-1} \in U_{r_u}$.

Using the triangle inequality, (B1) and Condition 3, leads to

$$\left\| h_{i+1}' \right\|_{U_{r_u}} \leq \left\| B^{-1} \right\| L + \left\| B^{-1} \right\| \cdot L \cdot \left\| h_{i+1}' \right\|_{U_{r_u}} + \left\| B^{-1} \right\| \left\| h_i' \right\|_{U_{r_u}} \cdot (\|A\| + L) + \left\| B^{-1} \right\| \left\| h_i' \right\|_{U_{r_u}} L \left\| h_{i+1}' \right\|_{U_{r_u}}$$

Transferring the terms containing $\left\| h_{i+1}' \right\|_{U_{r_u}}$ to the left-hand side, gives

$$\left\| h_{i+1}' \right\|_{U_{r_u}} \cdot (1 - \left\| B^{-1} \right\| L - \left\| B^{-1} \right\| L \left\| h_i' \right\|_{U_{r_u}}) \leq \left\| B^{-1} \right\| \cdot L + \left\| B^{-1} \right\| \left\| h_i' \right\|_{U_{r_u}} \cdot (\|A\| + L).$$

By the inductive assumption, $\left\| h_i' \right\|$ satisfies the inequality (B5); hence, $\left(1 - \left\| B^{-1} \right\| L - \left\| B^{-1} \right\| \left\| h_i' \right\| L \right) > 0$. Then it follows that

$$\left\| h_{i+1}' \right\|_{U_{r_u}} \leq \frac{\left\| B^{-1} \right\| L + \left\| B^{-1} \right\| \left\| h_i' \right\|_{U_{r_u}} \cdot (\|A\| + L)}{\left(1 - \left\| B^{-1} \right\| L - \left\| B^{-1} \right\| L \left\| h_i' \right\|_{U_{r_u}} \right)} \ \text{(B16)}.$$

Now consider the following difference equation:

$$s_{i+1} = \frac{\rho + \left(\left\|B^{-1}\right\| \cdot \|A\| + \rho\right)s_i}{\left(1 - \rho - \rho s_i\right)} \text{ (B17)}$$

where $\rho = \left\|B^{-1}\right\|L$.

Lemma

Suppose

$\rho < (1 - \left\|B^{-1}\right\| \cdot \|A\|)/4;$ (B18).

Then the difference equation (B17) has two fixed points:

$$s_1^* = \frac{1 - 2\rho - \left\|B^{-1}\right\| \cdot \|A\| - \sqrt{\left(1 - 2\rho - \left\|B^{-1}\right\| \cdot \|A\|\right)^2 - 4\rho^2}}{2\rho} \text{ (B19)}$$

and

$$s_2^* = \frac{1 - 2\rho - \left\|B^{-1}\right\| \cdot \|A\| + \sqrt{\left(1 - 2\rho - \left\|B^{-1}\right\| \cdot \|A\|\right)^2 - 4\rho^2}}{2\rho} \text{ (B20)}$$

such that

$$s_1^* \le s_2^* < \frac{1 - \rho}{\rho} \text{ (B21)}$$

where s_1^* is a stable fixed point, whereas s_2^* is an unstable one. If $s_0 = 0$, then s_i, $i = 1,2, ...$ is a monotonically increasing sequence that converges to s_1^*.

Proof

The lemma can be proved by direct calculation.

The inequality (B18) follows easily from Condition 2 of the theorem. Comparing (B16) and (B17) for the initial point $s_0 = 0$ and the initial mapping $h_0 \equiv 0$, we have

$s_i \ge \left\|h_i^{'}\right\|$, $i = 1,2,3, ...$, i.e. $\left\|h_i^{'}\right\|$ is majorised by s_i. Taking into account (B21), results in

$$\left\|h_{i+1}^{'}\right\| \le s_1^* \le \frac{1 - \rho}{\rho} = \frac{1 - \left\|B^{-1}\right\|L}{\left\|B^{-1}\right\|L}.$$

Therefore, the mapping h_{i+1} satisfies the inequality (B4). This concludes the induction argument and thus the proof of the Theorem.

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